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Exam FAM-S Study Manual

1st Edition, 3rd Printing

Abraham Weishaus, Ph.D., FSA, CFA, MAAA

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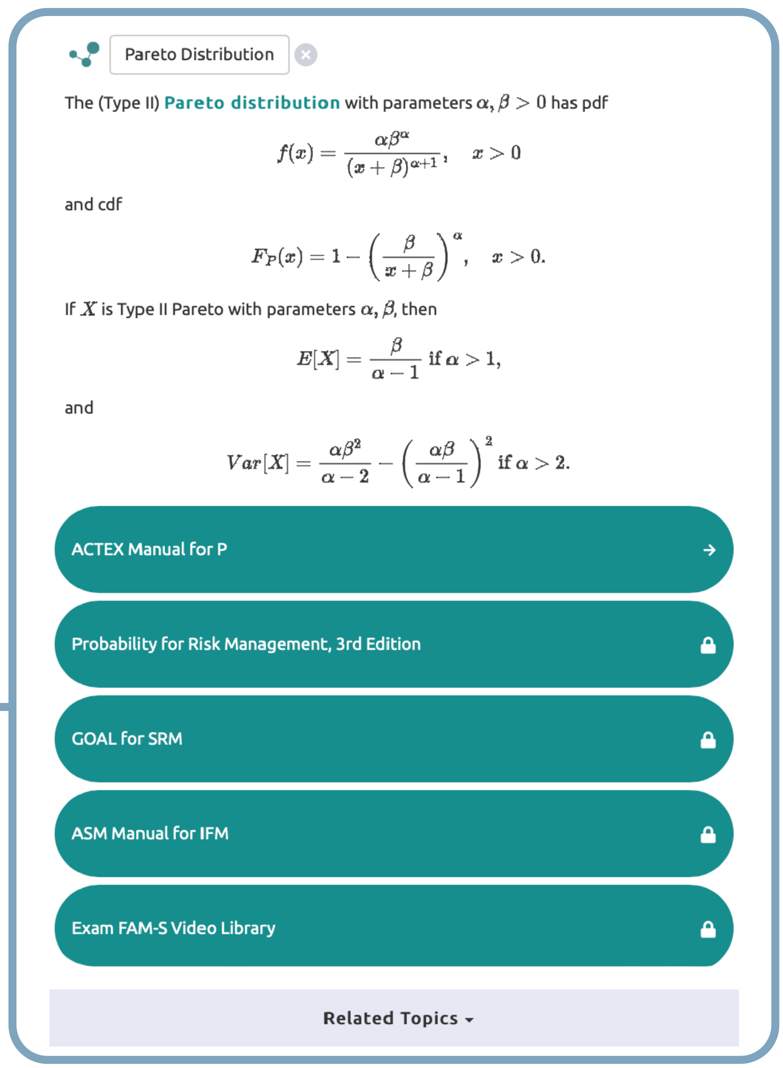

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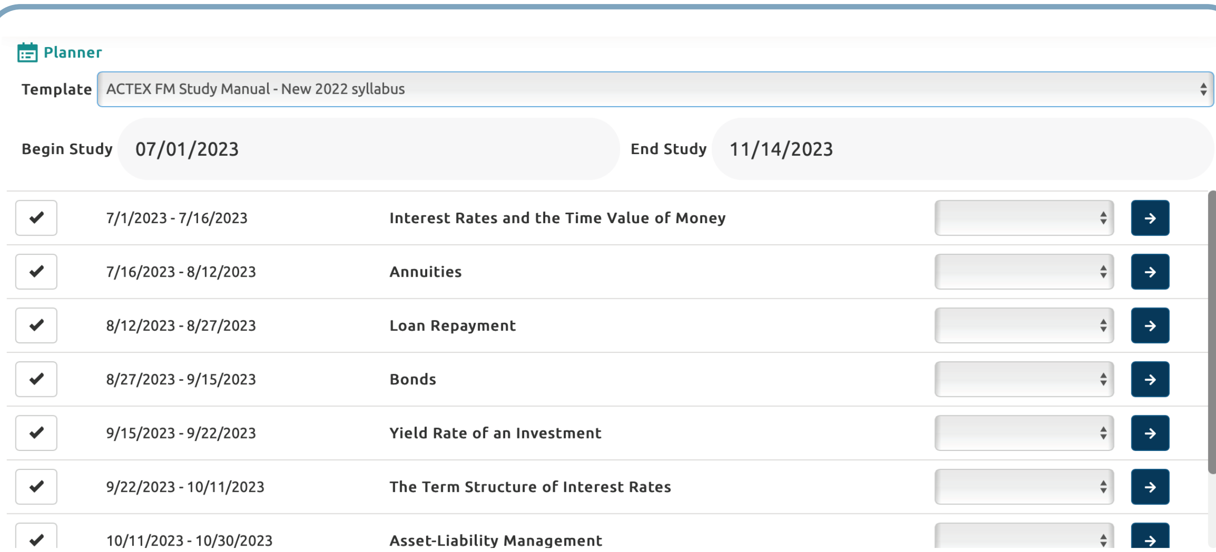

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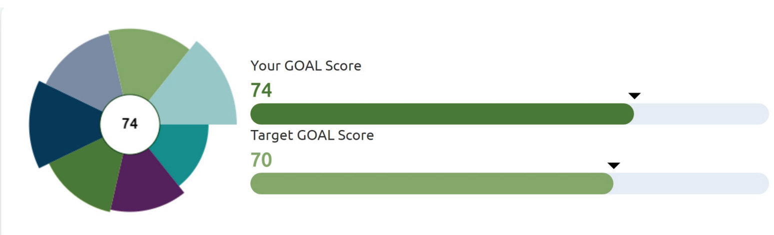

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64 A. Multiple State Models Weighting: 20%	71 B. Multiple Decrement Models Weighting: 10%	73 C. Multiple Life Models Weighting: 10%
82 D. Profit Analysis Weighting: 15%	76 E. Universal Life Insurance Weighting: 15%	78 F. Embedded Options and Equity-Linked Insurance Weighting: 15%
87 G. Pension Plans and Retirement Benefits Weighting: 15%		

See key areas where you can improve.

A. Multiple State Models — A1. Multiple State Probabilities

Difficulty	Mean	Proportion
Core	38	22 / 48
Advanced	38	18 / 48
Mastery	38	8 / 48

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Created	Last Accessed	Completed	Mode	Categories	Questions	Status	Action
03/24/2023	03/24/2023	03/24/2023	Quiz	G. Pension Plans and Retirement Benefits	15	Complete	Review
03/24/2023	03/24/2023	03/24/2023	Quiz	C1. Joint Life/Last Survivor Probabilities, C2. Contingent Probabilities, C3. Expectation and Variance, C6. ML Models with Transition Intensities, C4. ML Models: Premiums and Reserves (Discrete), C5. ML Models: Premiums and Reserves (Continuous)	15	Complete	Review
03/24/2023	03/24/2023	03/24/2023	Quiz	F. Embedded Options and Equity-Linked Insurance	20	Complete	Review
03/24/2023	03/24/2023	03/24/2023	Quiz	D. Profit Analysis	10	Complete	Review
09/19/2022	09/19/2022	09/19/2022	Quiz	G4. Pension Plans and Retirement Benefits (Written Answers)	2	Complete	Review

Quickly return to previous sessions.

Contents

Preface

Exam FAM-S discusses the mathematics of short-term insurance. Short-term insurance is insurance for periods of one year or less. For these lines of insurance, there is no guaranteed renewal, and even if the insurance is renewed from year to year, premium rates are usually updated every year. Also, there is no cost for switching insurers from one year to the next. This is unlike long-term insurance such as life insurance.

As a candidate for ASA and FSA, you probably won't be working in a property/casualty insurance company. And the typical insurance discussed in this course is auto insurance. But some of the methods we discuss are useful for medical and dental insurance, products sold by life insurance companies. And some concepts, such as credibility, are useful for mortality studies and reserving.

Prerequisites for most of the material are few beyond knowing probability (and calculus of course). Some elementary statistics will be helpful, especially for Part IV of the manual.

This manual

The exercises in this manual

I've provided lots of my own exercises, as well as relevant exercises from old exams. Though the style of exam questions has changed a little, these are still very useful practice exercises which cover the same material—don't dismiss them as obsolete!

All SOA or joint exam questions in this manual from exams given in 2000 and later, with solutions, are also available on the web from the SOA. When the 2000 syllabus was established in 1999, sample exams 3 and 4 were created, consisting partially of questions from older exams and partially of new questions, not all multiple choice. These sample exams were not real exams, and some questions were inappropriate or defective. These sample exams are no longer posted on the web. I have included appropriate questions, labeled "1999 C3 Sample" or "1999 C4 Sample". *These refer to these 1999 sample exams, not to more recent sets of sample questions that may be posted.*

Questions from old exams are marked xxx:yy, where xxx is the time the exam was given, with S for spring and F for fall followed by a 2-digit year, and yy is the question number. Sometimes xxx is preceded with SOA or CAS to indicate the sponsoring organization. From about 1986 to 2000, SOA exams had 3-digit numbers (like 160) and CAS exams were a number and a letter (like 4B). From 2000 to Spring 2003, exam 3 was jointly sponsored, so I do not indicate "SOA" or "CAS" for exam 3 questions from that period. There was a period in the 1990's when the SOA, while releasing old exam questions, did not indicate which exam they came from. As a result, I sometimes cannot identify the source exam for questions from this period. In such a case, I mark the question aaa-bb-cc:yy, where aaa-bb-cc is the study note number and yy is the question number. Generally aaa is the exam number (like 160), and cc is the 2-digit year the study note was published.

Index

This manual has an index. Whenever you remember some topic in this manual but can't remember where you saw it, check the index. If it isn't in the index but you're sure it's in the manual and an index listing would be appropriate, contact the author.

Flashcards

Many students find flashcards a useful tool for learning key formulas and concepts. ASM flashcards, available from the same distributors that sell this manual, contain the formulas and concepts from this manual in a convenient deck of cards. The cards have crossreferences, usually by page, to the manual.

Downloads from the SOA website

Tables

At the exam, you will be given distribution tables. They are available at the following link:

https://www.soa.org/49f99d/globalassets/assets/files/edu/2022/tables-fam-s.pdf

These give you moments and distribution functions for many distributions, so you need not memorize them. It also gives you a chi-square distribution table, which will have no use on this exam. (It would be useful on ASTAM, but they don't give it to you for that exam—they instead give you a worksheet that includes the chi-square function.)

One of the things that is missing from it, that used to be provided, is the normal distribution table. They will give you the Prometric calculator, which will calculate the standard normal distribution and its inverse to 5 decimal places. This precision is especially useful for option pricing, but doesn't make much difference in the rest of this course. While this manual usually works out the exercises with precision, I've used the traditional 3-digit approximations of the 90$^{\text{th}}$, 95$^{\text{th}}$, and 97.5$^{\text{th}}$ percentiles (1.282, 1.645, and 1.960 respectively) in my solutions.

Notation and terminology note

The notation and terminology note are at

https://www.soa.org/4a1b9f/globalassets/assets/files/edu/2022/fam-s-notation-note.pdf

This manual uses notation and terminology in accordance with that note.

Sample questions and solutions

The following links have 76 sample questions and solutions for this exam:

https://www.soa.org/49f968/globalassets/assets/files/edu/2022/fam-s-sample-questions.pdf
https://www.soa.org/49f992/globalassets/assets/files/edu/2022/fam-s-sample-solutions.pdf

To appreciate these questions, let's go through their history. During 2000–Spring 2007, the SOA released associateship exams 9 times. That was the pre-CBT era. Later on, the SOA compiled a list of about 300 sample questions based on 6 of those exams; they did not use the two 2000 administrations and did not use the Spring 2007 exam. The syllabus of C did not change much; a few questions were deleted as their topics dropped from the syllabus, but nothing was added for the few new topics. In 2018, when Exam C became Exam STAM, insurance coverages were added to the syllabus, and 22 additional sample questions were added.

The 76 sample questions you have were selected from the old list; nothing was added. They did not include every question that was relevant. But strangely—probably by error—they included 6 questions not covered by the syllabus reading, questions on topics assigned to ASTAM. Refer to Table B.2, which has and "NS" indicator for those questions. They edited the questions slightly, but no material changes were made. This manual does not include those 6 questions, but does include the others, as well as other old exam questions not in that list.

As a result of the way the list was compiled, the number of questions on each topic is not in accordance with syllabus weights. More importantly, no questions are provided on topics added to FAM-S that were not on STAM, namely option pricing.

Notes about the exam

Topic weights

The syllabus breaks the course down into 6 topics. Their weights[1] and the lessons in the manual that cover them are:

[1]The weights listed in the syllabus are for the full FAM exam. I've doubled the weights so that they represent the weights for FAM-S.

Table 1: Nine Week Study Schedule for Exam FAM-S

Week	Subject	Lessons	Rarely Tested
1	Probability basics	1–4	2.3, 2.4
2	Short term insurances and loss reserves	5–7	
3	Ratemaking	8–9	
4	Severity modifications	10–12	
5	Reinsurance, risk measures, and frequency	13–18	14.4, 14.5, *16*, 17.2
6	Aggregate loss	19–22	*19.3*
7	Maximum likelihood	23–25	
8	Credibility	26–28	27
9	Option Pricing	29–31	

Topic	Weight	Lessons
1. Insurance and Reinsurance Coverages	15–25%	5–6, 10–13, 15–16
2. Severity, Frequency, and Aggregate Models	25–30%	1–4, 14, 17–22
3. Parametric and Non-Parametric Estimation	10–20%	23–25
4. Introduction to Credibility	5–10%	26–28
5. Pricing and Reserving for Short-Term Insurance Coverages	15–25%	7–9
6. Option Pricing Fundamentals	5–15%	29–31

Since the exam has 20 questions, each 5% is one question.

Guessing penalty

There is no guessing penalty on this exam. So fill in every answer—you may be lucky! Leave yourself a couple of seconds to do this.

Calculators

A wide variety of calculators are permitted: the TI-30Xa, TI-30X II battery or solar, TI-30X MultiView battery or solar, the BA-35 (battery or solar), and the BA-II Plus (or BA II Plus Professional Edition). You may bring several calculators into the exam. The MultiView calculator is considered the best one, due to its data tables which allow fast statistical calculations. The data table is a very restricted spreadsheet. Despite its limitations, it is useful.

Another feature of the Multiview is storage of previous calculations. They can be recalled and edited.

Other features which may be of use are the K constant and the table feature, which allows calculation of a function at selected values or at values in an arithmetic progression.

Financial calculations do not occur on this exam; interest is almost never considered. You will not miss the lack of financial functions on the Multiview.

Study Schedule

Different students will have different speeds and different constraints, so it's hard to create a study schedule useful for everybody. However, I offer a sample 9-week study schedule, Table 1, as a guide. The last column lists rarely tested materials so you can skip those if you are behind in your schedule. Italicized sections in this column are, in my opinion, extremely unlikely exam topics.

Errata

Please report any errors you find. Reports may be sent to the publisher (mail@studymanuals.com) or directly to me (errata@aceyourexams.net). *When reporting errata, please indicate which manual and which edition and printing you are referring to!* This manual is the 1st edition 3rd printing of the Exam FAM-S manual.

An errata list will be posted at http://errata.aceyourexams.net

Acknowledgements

I wish to thank the Society of Actuaries and the Casualty Actuarial Society for permission to use their old exam questions. These questions are the backbone of this manual.

I wish to thank Donald Knuth, the creator of TEX, Leslie Lamport, the creator of LATEX, and the many package writers and maintainers, for providing a typesetting system that allows such beautiful typesetting of mathematics and figures.

I wish to thank Darnelia Baugh, Gregory Mandell, and Rebecca Sheehan for the errata they sent in.

Part I

Probability and Insurance Coverages

Lesson 2

Parametric Distributions

Reading: *Loss Models* Fifth Edition 4, 5.3—5.4

A ***parametric distribution*** is one that is defined by a fixed number of parameters. Examples of parametric distributions are the **exponential distribution** (parameter θ) and the **Pareto distribution** (parameters α, θ). Any distribution listed in the *Loss Models* appendix is parametric.

The alternative to a parametric distribution is a ***data-dependent distribution***. A data-dependent distribution is one where the specification requires at least as many "parameters" as the number of data points in the sample used to create it; the bigger the sample, the more "parameters".

It is traditional to use parametric distributions for claim counts (**frequency**) and loss size (**severity**). Parametric distributions have many advantages. One of the advantages of parametric distributions which makes them so useful for severity is that they handle inflation easily.

2.1 Scaling

A parametric distribution is a member of a ***scale family*** if any positive multiple of the random variable has the same form. In other words, the distribution function of cX, for c a positive constant, is of the same form as the distribution function of X, but with different values for the parameters. Sometimes the distribution can be parametrized in such a way that only one parameter of cX has a value different from the parameters of X. If the distribution is parametrized in this fashion, so that the only parameter of cX having a different value from X is θ, and the value of θ for cX is c times the value of θ for X, then θ is called a ***scale parameter***.

All of the continuous distributions in the tables (Appendix A) are scale families. The parametrizations given in the tables are often different from those you would find in other sources, such as your probability textbook. They are parametrized so that θ is the scale parameter. Thus when you are given that a random variable has any distribution in the appendix and you are given the parameters, it is easy to determine the distribution of a multiple of the random variable.

The only distributions not parametrized with a scale parameter are the **lognormal** and the **inverse Gaussian**. Even though the inverse Gaussian has θ as a parameter, it is not a scale parameter. The parametrization for the lognormal given in the tables is the traditional one. *If you need to **scale a lognormal**, proceed as follows: if X is lognormal with parameters (μ, σ), then cX is lognormal with parameters $(\mu + \ln c, \sigma)$.*

To scale a random variable not in the tables, you'd reason as follows. Let $Y = cX$, $c > 0$. Then

$$F_Y(y) = \Pr(Y \le y) = \Pr(cX \le y) = \Pr\left(X \le \frac{y}{c}\right) = F_X\left(\frac{y}{c}\right)$$

One use of scaling is in handling inflation. In fact, handling inflation is the only topic in this lesson that is commonly tested directly. If loss sizes are inflated by $100r\%$, the **inflated loss variable** Y will be $(1 + r)X$, where X is the pre-inflation loss variable. For a scale family with a scale parameter, you just multiply θ by $(1 + r)$ to obtain the new distribution.

Example 2A Claim sizes expressed in dollars follow a two-parameter Pareto distribution with parameters $\alpha = 5$ and $\theta = 90$. A euro is worth \$1.50.

Calculate the probability that a claim will be for 20 euros or less. ■

Solution: If claim sizes in dollars are X, then claim sizes in euros are $Y = X/1.5$. The resulting euro-based random variable Y for claim size will be Pareto with $\alpha = 5$, $\theta = 90/1.5 = 60$. The probability that a claim will be no more than 20 euros is

$$\Pr(Y \le 20) = F_Y(20) = 1 - \left(\frac{60}{60+20}\right)^5 = \boxed{\mathbf{0.7627}}$$

□

Example 2B Claim sizes in 2021 follow a lognormal distribution with parameters $\mu = 4.5$ and $\sigma = 2$. Claim sizes grow at 6% uniform inflation during 2022 and 2023.

Calculate $f(1000)$, the probability density function at 1000, of the claim size distribution in 2023. ■

Solution: If X is the claim size random variable in 2021, then $Y = 1.06^2X$ is the revised variable in 2023. The revised lognormal distribution of Y has parameters $\mu = 4.5 + 2\ln 1.06$ and $\sigma = 2$. The probability density function at 1000 is

$$\begin{aligned} f_Y(1000) &= \frac{1}{\sigma(1000)\sqrt{2\pi}}e^{-(\ln 1000-\mu)^2/2\sigma^2} \\ &= \frac{1}{(2)(1000)\sqrt{2\pi}}e^{-[\ln 1000-(4.5+2\ln 1.06)]^2/2(2^2)} \\ &= (0.000199471)(0.518814) = \boxed{\mathbf{0.0001035}} \end{aligned}$$

□

Example 2C Claim sizes expressed in dollars follow a lognormal distribution with parameters $\mu = 3$ and $\sigma = 2$. A euro is worth \$1.50.

Calculate the probability that a claim will be for 100 euros or less. ■

Solution: If claim sizes in dollars are X, then claim sizes in euros are $Y = X/1.5$. As discussed above, the distribution of claim sizes in euros is lognormal with parameters $\mu = 3 - \ln 1.5$ and $\sigma = 2$. Then

$$F_Y(100) = \Phi\left(\frac{\ln 100 - 3 + \ln 1.5}{2}\right) = \Phi(1.01) = \boxed{\mathbf{0.8438}}$$

□

Example 2D Claim sizes X initially follow a distribution with distribution function:

$$F_X(x) = 1 - \frac{1}{e^{0.01x}(1+0.01x)} \qquad x > 0$$

Claim sizes are inflated by 50% uniformly.

Calculate the probability that a claim will be for 60 or less after inflation. ■

Solution: Let Y be the increased claim size. Then $Y = 1.5X$, so $\Pr(Y \le 60) = \Pr(X \le 60/1.5) = F_X(40)$.

$$F_X(40) = 1 - \frac{1}{1.4e^{0.4}} = \boxed{\mathbf{0.5212}}$$

□

2.2 Common parametric distributions

The tables provide a lot of information about the distributions, but if you don't recognize the distribution, you won't know to use the table. Therefore, it is a good idea to be familiar with the common distributions.

You should familiarize yourself with the *form* of each distribution, but not necessarily the constants. The constant is forced so that the density function will integrate to 1. If you know which distribution you are dealing with, you can figure out the constant. To emphasize this point, in the following discussion, we will use the letter c for constants rather than spelling out what the constants are. You are not trying to recognize the constant; you are trying to recognize the form.

We will mention the means and variances or second moments of the distributions. You need not memorize any of these. The tables give you the **raw moments**. You can calculate the **variance** as $\mathbf{E}[X^2] - \mathbf{E}[X]^2$. However, for

The gamma function

The gamma function $\Gamma(x)$ is a generalization to real numbers of the factorial function, defined by

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} \mathrm{d}u$$

For positive integers n,

$$\Gamma(n) = (n-1)!$$

The most important relationship for $\Gamma(x)$ that you should know is

$$\Gamma(x+1) = x\Gamma(x)$$

for any real number x.

EXAMPLE 2E Evaluate $\dfrac{\Gamma(8.5)}{\Gamma(6.5)}$.

SOLUTION:

$$\frac{\Gamma(8.5)}{\Gamma(6.5)} = \left(\frac{\Gamma(8.5)}{\Gamma(7.5)}\right)\left(\frac{\Gamma(7.5)}{\Gamma(6.5)}\right) = (7.5)(6.5) = \boxed{\mathbf{48.75}}$$

---

frequently used distributions, you may want to memorize the mean and variance to save yourself some time when working out questions.

We will graph the distributions. You are not responsible for graphs, but they may help you understand the distributions.

The tables occasionally use the **gamma function** $\Gamma(x)$ in the formulas for the moments. You should have a basic knowledge of the gamma function; if you are not familiar with this function, see the sidebar. The tables also use the **incomplete gamma** and **beta functions**, and define them, but you can get by without knowing them.

2.2.1 Uniform

A **uniform distribution** has a constant density on $[d, u]$:

$$f(x; d, u) = \frac{1}{u-d} \qquad d \le x \le u$$

$$F(x; d, u) = \begin{cases} 0 & x \le d \\ \dfrac{x-d}{u-d} & d \le x \le u \\ 1 & x \ge u \end{cases}$$

You recognize a uniform distribution both by its finite **support**[1] and by the lack of an x in the density function.

Its moments are

$$\mathbf{E}[X] = \frac{d+u}{2}$$

$$\mathrm{Var}(X) = \frac{(u-d)^2}{12}$$

[1]"Support" is the range of values for which the probability density function is nonzero.

Its mean, median, and midrange are equal. The best way to calculate the second moment is to add up the variance and the square of the mean. However, some students prefer to use the following easy-to-derive formula:

$$\mathbf{E}[X^2] = \frac{1}{u-d}\int_d^u x^2\,\mathrm{d}x = \frac{u^3-d^3}{3(u-d)} = \frac{u^2+ud+d^2}{3} \tag{2.1}$$

If $d = 0$, then the formula reduces to $u^2/3$.

The uniform distribution is not directly in the tables, so I recommend you memorize the formulas for mean and variance. However, if $d = 0$, then the uniform distribution is a special case of a beta distribution with $\theta = u$, $a = 1$, $b = 1$.

2.2.2 Beta

The probability density function of a **beta distribution** with $\theta = 1$ has the form

$$f(x; a, b) = cx^{a-1}(1-x)^{b-1} \qquad 0 \le x \le 1$$

The parameters a and b must be positive. They may equal 1, in which case the corresponding factor is missing from the density function. Thus if $a = b = 1$, the beta distribution is a uniform distribution.

You recognize a beta distribution both by its finite support—it's the only common distribution for which the density is nonzero only on a finite range of values—and by factors with x and $1-x$ raised to powers and no other use of x in the density function.

If θ is arbitrary, then the form of the probability density function is

$$f(x; a, b, \theta) = cx^{a-1}(\theta-x)^{b-1} \qquad 0 \le x \le \theta$$

The distribution function can be evaluated if a or b is an integer. The moments are

$$\mathbf{E}[X] = \frac{\theta a}{a+b}$$
$$\mathrm{Var}(X) = \frac{\theta^2 ab}{(a+b)^2(a+b+1)}$$

The mode is $\theta(a-1)/(a+b-2)$ when a and b are both greater than 1, but you are not responsible for this fact.

Figure 2.1 graphs four beta distributions with $\theta = 1$ all having mean $2/3$. You can see how the distribution becomes more peaked and normal looking as a and b increase.

2.2.3 Exponential

The probability density function of an **exponential distribution** has the form

$$f(x; \theta) = ce^{-x/\theta} \qquad x \ge 0$$

θ must be positive.

You recognize an exponential distribution when the density function has e raised to a multiple of x, and no other use of x.

The distribution function is easily evaluated. The moments are:

$$\mathbf{E}[X] = \theta$$
$$\mathrm{Var}(X) = \theta^2$$

Figure 2.2 graphs three exponential distributions. The higher the parameter, the more weight placed on higher numbers.

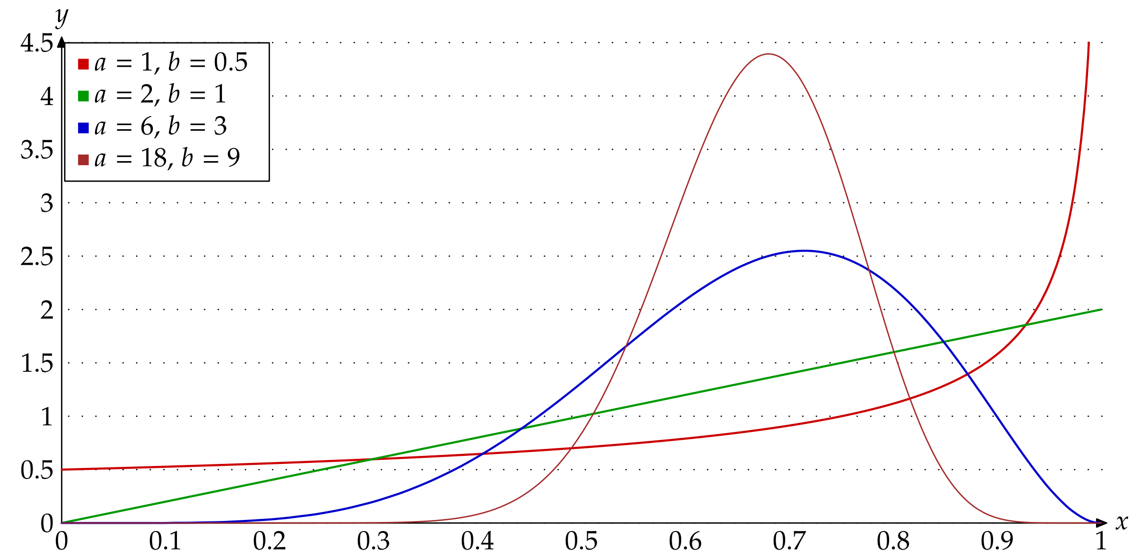

Figure 2.1: Probability density function of four beta distributions with $\theta = 1$ and mean 2/3

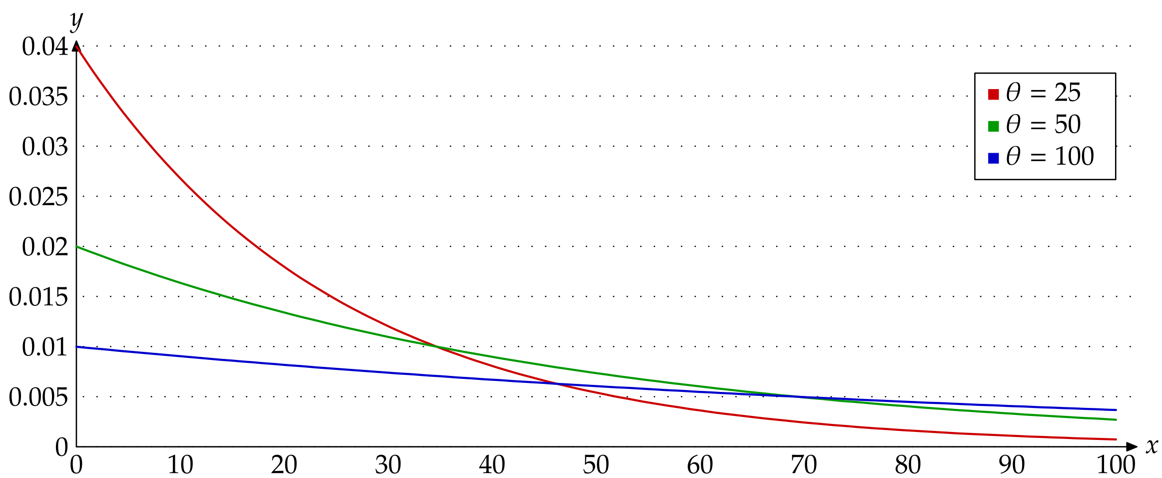

Figure 2.2: Probability density function of three exponential distributions

2.2.4 Weibull

A **Weibull distribution** is a transformed exponential distribution. If Y is exponential with mean μ, then $X = Y^{1/\tau}$ is Weibull with parameters $\theta = \mu^{1/\tau}$ and τ. An exponential is a special case of a Weibull with $\tau = 1$.

The form of the density function is

$$f(x;\tau,\theta) = cx^{\tau-1}e^{-(x/\theta)^{\tau}} \qquad x \geq 0$$

Both parameters must be positive.

You recognize a Weibull distribution when the density function has e raised to a multiple of a power of x, and in addition has a corresponding power of x, one lower than the power in the exponential, as a factor.

The distribution function is easily evaluated, but the moments require evaluating the **gamma function**, which usually requires numerical techniques. The moments are

$$\mathbf{E}[X] = \theta\Gamma(1 + 1/\tau)$$
$$\mathbf{E}[X^2] = \theta^2\Gamma(1 + 2/\tau)$$

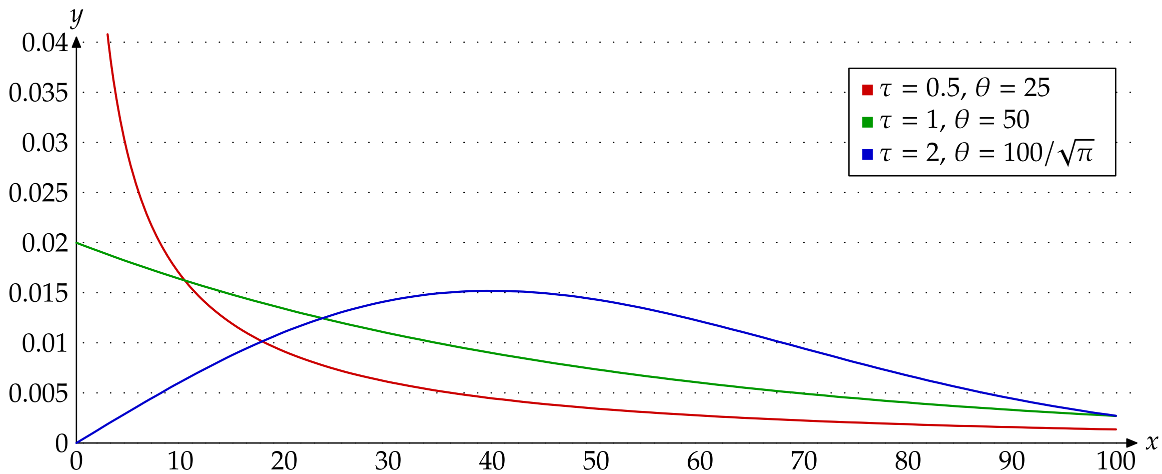

Figure 2.3: Probability density function of three Weibull distributions with mean 50

Figure 2.3 graphs three Weibull distributions with mean 50. The distribution has a non-zero mode when $\tau > 1$. Notice that the distribution with $\tau = 0.5$ puts a lot of weight on small numbers. To make up for this, it will also have to put higher weight than the other two distributions on very large numbers, so although it's not shown, its graph will cross the other two graphs for high x

2.2.5 Gamma

The form of the density function of a **gamma distribution** is

$$f(x;\alpha,\theta) = cx^{\alpha-1}e^{-x/\theta} \qquad x \ge 0$$

Both parameters must be positive.

When α is an integer, a gamma random variable with parameters α and θ is the sum of α independent exponential random variables with parameter θ. In particular, when $\alpha = 1$, the gamma random variable is exponential. The gamma distribution is called an **Erlang distribution** when α is an integer.

You recognize a gamma distribution when the density function has e raised to a multiple of x, and in addition has x raised to a power. Contrast this with a Weibull, where e is raised to a multiple of a *power* of x.

The distribution function may be evaluated if α is an integer; otherwise numerical techniques are needed. However, the moments are easily evaluated:

$$\mathbf{E}[X] = \alpha\theta$$
$$\mathrm{Var}(X) = \alpha\theta^2$$

Figure 2.4 graphs three gamma distributions with mean 50. As α goes to infinity, the graph's peak narrows and the distribution converges to a normal distribution.

The gamma distribution is one of the few for which the moment generating function has a closed form. In particular, the moment generating function of an exponential has a closed form. The only other distributions in the tables with closed form moment generating functions are the normal distribution (not actually in the tables, but the formula for the lognormal moments is the MGF of a normal) and the inverse Gaussian.

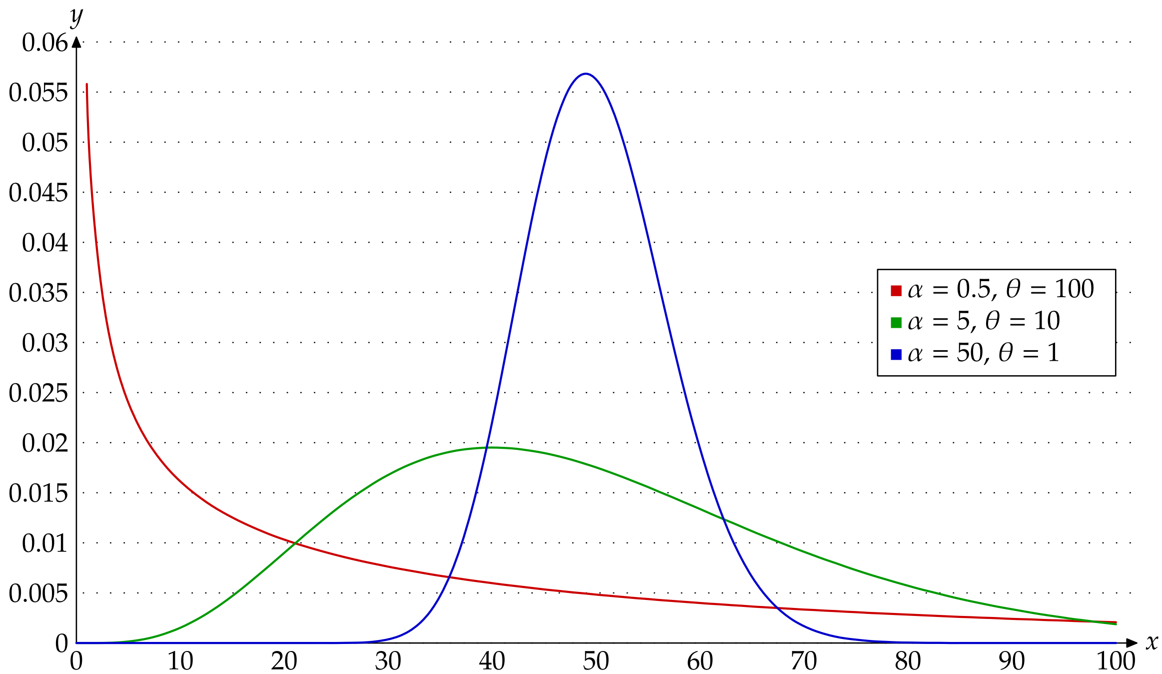

Figure 2.4: Probability density function of three gamma distributions with mean 50

2.2.6 Pareto

When we say "Pareto", we mean a *two-parameter* Pareto. On recent exams, they write out "two-parameter" to make it clear, but on older exams, you will often find the word "Pareto" with no qualifier. It always refers to a two-parameter Pareto, not a single-parameter Pareto.

The form of the density function of a **two-parameter Pareto** is

$$f(x) = \frac{c}{(\theta + x)^{\alpha+1}} \qquad x \geq 0$$

Both parameters must be positive.

You recognize a Pareto when the density function has a denominator with x plus a constant raised to a power.

The distribution function is easily evaluated. The moments are

$$\mathbf{E}[X] = \frac{\theta}{\alpha - 1} \qquad \alpha > 1$$

$$\mathbf{E}[X^2] = \frac{2\theta^2}{(\alpha-1)(\alpha-2)} \qquad \alpha > 2$$

When α does not satisfy these conditions, the corresponding moments don't exist.

A shortcut formula for the variance of a Pareto is

$$\operatorname{Var}(X) = \mathbf{E}[X]^2\left(\frac{\alpha}{\alpha - 2}\right)$$

Figure 2.5 graphs three Pareto distributions, one with $\alpha < 1$ and the other two with mean 50. Although the one with $\alpha = 0.5$ puts higher weight on small numbers than the other two, its mean is infinite; it puts higher weight on large numbers than the other two, and its graph eventually crosses the other two as $x \to \infty$.

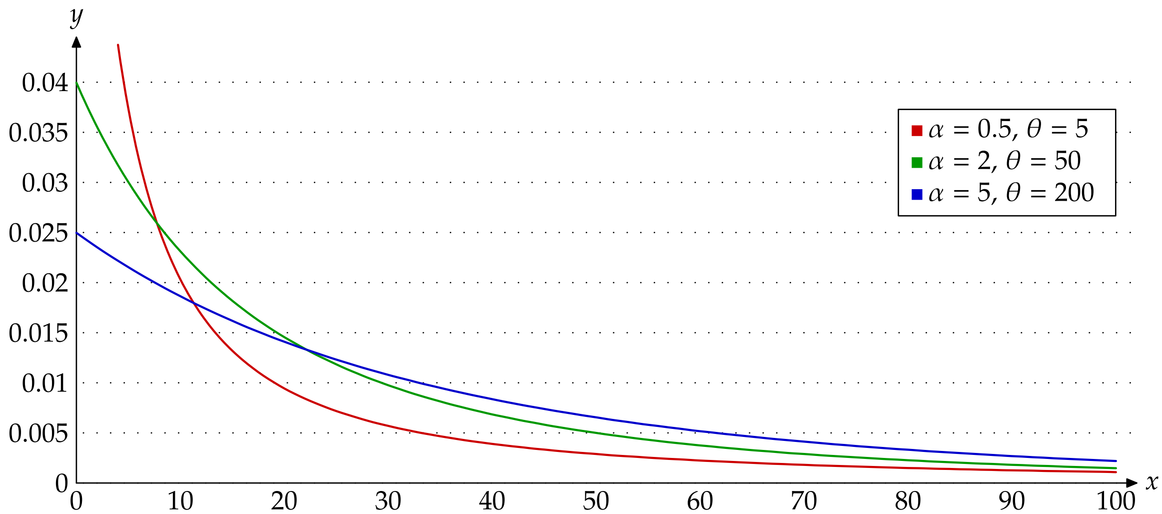

Figure 2.5: Probability density function of three Pareto distributions

2.2.7 Single-parameter Pareto

The form of the density function of a **single-parameter Pareto** is

$$f(x) = \frac{c}{x^{\alpha+1}} \qquad x \geq \theta$$

α must be positive. θ is not considered a parameter since it must be selected in advance, based on what you want the range to be.

You recognize a single-parameter Pareto by the range of nonzero values for its density function—unlike most other distributions, this range does not start at 0—and by the form of the density function, which has a denominator with x raised to a power. A beta distribution may also have x raised to a negative power, but its density function is 0 above a finite number.

A single-parameter Pareto X is a two-parameter Pareto Y shifted by θ: $X = Y + \theta$. Thus it has the same variance, and the mean is θ greater than the mean of a two-parameter Pareto with the same parameters.

$$\mathbf{E}[X] = \frac{\alpha\theta}{\alpha - 1} \qquad \alpha > 1$$
$$\mathbf{E}[X^2] = \frac{\alpha\theta^2}{\alpha - 2} \qquad \alpha > 2$$

2.2.8 Lognormal

The form of the density function of a **lognormal distribution** is

$$f(x) = \frac{ce^{-(\ln x - \mu)^2/2\sigma^2}}{x} \qquad x > 0$$

σ must be nonnegative.

You recognize a lognormal by the $\ln x$ in the exponent.

If Y is normal, then $X = e^Y$ is lognormal with the same parameters μ and σ. Thus, to calculate the distribution function, use

$$F_X(x) = F_Y(\ln x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$$

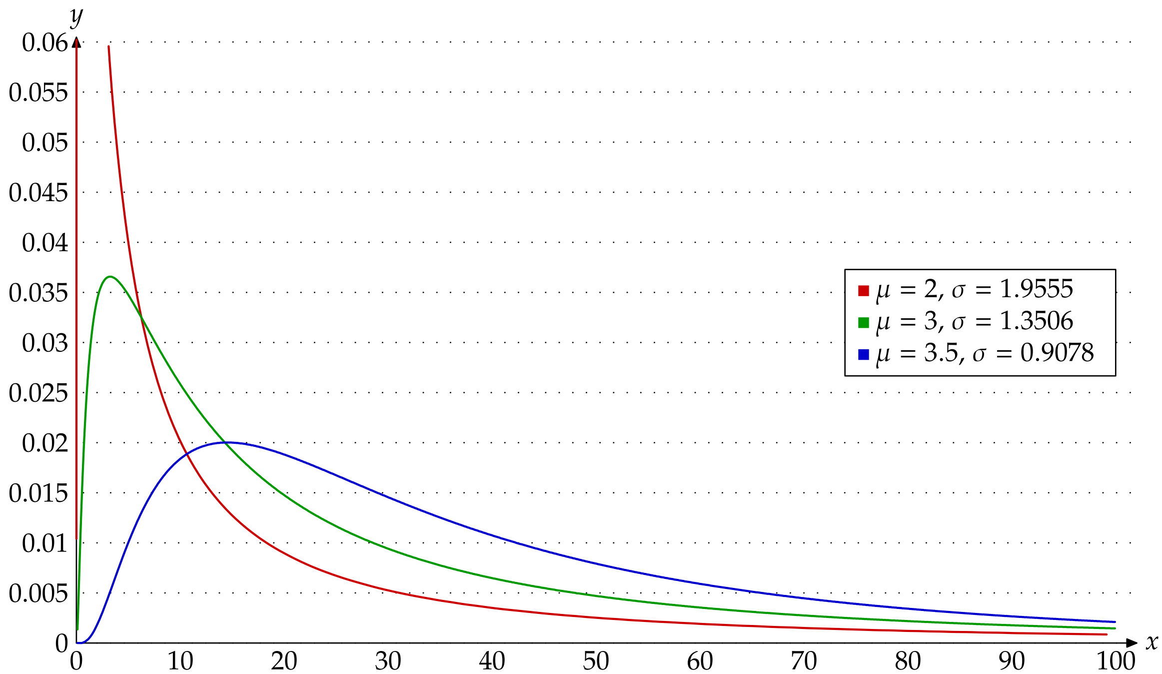

Figure 2.6: Probability density function of three lognormal distributions with mean 50

where $\Phi(x)$ is the standard normal distribution function, for which you are given tables. The moments of a lognormal are

$$\mathbf{E}[X] = e^{\mu+0.5\sigma^2}$$
$$\mathbf{E}[X^2] = e^{2\mu+2\sigma^2}$$

More generally, $\mathbf{E}[X^k] = \mathbf{E}[e^{kY}] = M_Y(k)$, where $M_Y(k)$ is the moment generating function of the corresponding normal distribution.

Figure 2.6 graphs three lognormals with mean 50. The mode is $\exp(\mu - \sigma^2)$, as stated in the tables. For $\mu = 2$, the mode is off the graph. As σ gets lower, the distribution flattens out.

Table 2.1 is a summary of the forms of probability density functions for common distributions.

2.3 The linear exponential family

The following material is based on *Loss Models* 5.4 which is on the syllabus. It won't play any role in this course, but the **linear exponential family** is commonly used for generalized linear models[2], which you'll study when working on Exam SRM. I doubt anything in this section will be tested on directly, so you may skip it.

A set of parametric distributions is in the linear exponential family if it can be parametrized with a parameter θ in such a way that in its density function, the only interaction between θ and x is in the exponent of e, which is x times a function of θ. In other words, its density function $f(x;\theta)$ can be expressed as

$$f(x;\theta) = \frac{p(x)e^{r(\theta)x}}{q(\theta)}$$

The set may have other parameters. $q(\theta)$ is the normalizing constant which makes the integral of f equal to 1. $r(\theta)$ is called the *canonical parameter* of the distribution.

[2]Many textbooks leave out "linear" and just call it the exponential family.

Table 2.1: Forms of probability density functions for common distributions

Distribution	Probability density function	
Uniform	c	$d \le x \le u$
Beta	$cx^{a-1}(\theta - x)^{b-1}$	$0 \le x \le \theta$
Exponential	$ce^{-x/\theta}$	$x \ge 0$
Weibull	$cx^{\tau-1}e^{-x^\tau/\theta^\tau}$	$x \ge 0$
Gamma	$cx^{\alpha-1}e^{-x/\theta}$	$x \ge 0$
Pareto	$\dfrac{c}{(x+\theta)^{\alpha+1}}$	$x \ge 0$
Single-parameter Pareto	$\dfrac{c}{x^{\alpha+1}}$	$x \ge \theta$
Lognormal	$\dfrac{ce^{-(\ln x-\mu)^2/2\sigma^2}}{x}$	$x > 0$

Examples of the linear exponential family are:

Gamma distribution The pdf is

$$f(x;\mu,\sigma) = \frac{x^{\alpha-1}e^{-x/\theta}}{\Gamma(\alpha)\theta^\alpha}$$

Let $r(\theta) = -1/\theta$, $p(x) = x^{\alpha-1}$, and $q(\theta) = \Gamma(\alpha)\theta^\alpha$.

Normal distribution The pdf is

$$f(x;\theta) = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$$

Let $\theta = \mu$. The denominator of the pdf does not have x or θ so it can go into $q(\theta)$ or into $p(x)$. The exponent can be expanded into

$$-\frac{x^2}{2\sigma^2} + \frac{x\theta}{\sigma^2} - \frac{\theta^2}{2\sigma^2}$$

and only the second summand involves both x and θ, and x appears to the first power. Thus we can set $p(x) = e^{-x^2/2\sigma^2}$, $r(\theta) = \theta/\sigma^2$, and $q(\theta) = e^{\theta^2/2\sigma^2}\sigma\sqrt{2\pi}$.

Discrete distributions are in the linear exponential family if we can express the probability function in the linear exponential form.

Poisson distribution For a Poisson distribution, the probability function is

$$f(x;\lambda) = e^{-\lambda}\frac{\lambda^x}{x!} = e^{-\lambda}\frac{e^{x\ln\lambda}}{x!}$$

We can let $\theta = \lambda$, and then $p(x) = 1/x!$, $r(\theta) = \ln\theta$, and $q(\theta) = e^\theta$.

The textbook develops the following formulas for the mean and variance of a distribution from the linear exponential family:

$$\begin{aligned}
\mathbf{E}[X] = \mu(\theta) &= \frac{q'(\theta)}{r'(\theta)q(\theta)} = \frac{\big(\ln q(\theta)\big)'}{r'(\theta)} \\
\operatorname{Var}(X) = v(\theta) &= \frac{\mu'(\theta)}{r'(\theta)}
\end{aligned}$$

Thus, in the above examples:

Gamma distribution

$$\begin{aligned}
\frac{\mathrm{d}\ln q}{\mathrm{d}\theta} &= \frac{\alpha}{\theta} \\
\frac{\mathrm{d}r}{\mathrm{d}\theta} &= \frac{1}{\theta^2} \\
\mathbf{E}[X] &= \frac{\alpha/\theta}{1/\theta^2} = \alpha\theta \\
\operatorname{Var}(X) &= \frac{\alpha}{1/\theta^2} = \alpha\theta^2
\end{aligned}$$

Normal distribution

$$\begin{aligned}
\big(\ln q(\theta)\big)' &= \frac{2\theta}{2\sigma^2} = \frac{\theta}{\sigma^2} \\
r'(\theta) &= \frac{1}{\sigma^2} \\
\mathbf{E}[X] &= \frac{\theta/\sigma^2}{1/\sigma^2} = \theta \\
\operatorname{Var}(X) &= \frac{1}{1/\sigma^2} = \sigma^2
\end{aligned}$$

Poisson distribution

$$\begin{aligned}
\big(\ln q(\theta)\big)' &= 1 \\
r'(\theta) &= \frac{1}{\theta} \\
\mathbf{E}[X] &= \frac{1}{1/\theta} = \theta \\
\operatorname{Var}(X) &= \frac{1}{1/\theta} = \theta
\end{aligned}$$

2.4 Limiting distributions

The following material is based on *Loss Models* 5.3.3. I don't think it has ever appeared on the exam and doubt it ever will.

In some cases, as the parameters of a distribution go to infinity, the distribution converges to another distribution. To demonstrate this, we will usually have to use the identity

$$\lim_{\alpha\to\infty}\left(1+\frac{r}{\alpha}\right)^{\alpha} = e^r$$

Equivalently, if c is a constant (not dependent on α), then

$$\lim_{\alpha\to\infty}\left(1+\frac{r}{\alpha}\right)^{\alpha+c}=\lim_{\alpha\to\infty}\left(1+\frac{r}{\alpha}\right)^{\alpha}\left(1+\frac{r}{\alpha}\right)^{c}=e^r$$

As a simple example (not in the textbook) of a limiting distribution, consider a **gamma distribution** with a fixed mean μ, and let $\alpha\to\infty$. Then $\theta=\mu/\alpha$. The **moment generating function** is

$$M(t)=(1-\theta t)^{-\alpha}=\frac{1}{\left(1-\frac{\mu t}{\alpha}\right)^{\alpha}}$$

and as $\alpha\to\infty$, the denominator goes to $e^{-\mu t}$, so $M(t)\to e^{\mu t}$, which is the moment generating function of the constant μ. So as $\alpha\to\infty$, the limiting distribution of a gamma is a distribution equal to the mean with probability 1.

As another example, let's carry out textbook exercise 5.21, which asks you to demonstrate that the limiting distribution of a **Pareto** with θ/α constant as $\alpha\to\infty$ is an exponential. Let $k=\theta/\alpha$. The density function of a Pareto is

$$\begin{aligned}f(x;\alpha,\theta)&=\frac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}=\frac{\alpha(\alpha k)^\alpha}{(\alpha k+x)^{\alpha+1}}\\&=\frac{k^\alpha}{(k+x/\alpha)^{\alpha+1}}=\frac{1}{k\big(1+(x/k)/\alpha\big)^{\alpha+1}}\end{aligned}$$

and the limit as $\alpha\to\infty$ is $(1/k)e^{-x/k}$. That is the density function of an exponential with mean k. Notice that as $\alpha\to\infty$, the mean of the Pareto converges to k.

Table 2.2: Summary of Parametric Distribution Concepts

- If X is a member of a scale family with scale parameter θ with value s, then cX is in the same family and has the same parameter values as X except that the scale parameter θ has value cs.
- All distributions in the tables are scale families with scale parameter θ except for lognormal and inverse Gaussian.
- If X is lognormal with parameters μ and σ, then cX is lognormal with parameters $\mu+\ln c$ and σ.
- If X is normal with parameters μ and σ^2, then e^X is lognormal with parameters μ and σ.
- See Table 2.1 to learn the forms of commonly occurring distributions. Useful facts are

Uniform on [d,u]	$\mathbf{E}[X]=\dfrac{d+u}{2}$
	$\mathrm{Var}(X)=\dfrac{(u-d)^2}{12}$
Uniform on [0,u]	$\mathbf{E}[X^2]=\dfrac{u^2}{3}$
Gamma	$\mathrm{Var}(X)=\alpha\theta^2$

- If Y is single-parameter Pareto with parameters α and θ, then $Y-\theta$ is two-parameter Pareto with the same parameters.
- X is in the linear exponential family if its probability density function can be expressed as

$$f(x;\theta)=\frac{p(x)e^{r(\theta)x}}{q(\theta)}$$

Exercises

2.1. Loss sizes for an insurance coverage follow an inverse gamma distribution with mean 6 and mode 4.

Calculate the coefficient of skewness for the losses.

(A) 3.1 (B) 3.2 (C) 3.3 (D) 3.4 (E) 3.5

2.2. For a commercial fire coverage

- In 2022, loss sizes follow a two-parameter Pareto distribution with parameters $\alpha = 4$ and θ.
- In 2023, there is uniform inflation at rate r.
- The 65$^{\text{th}}$ percentile of loss size in 2023 equals the mean loss size in 2022.

Determine r.

2.3. **[CAS3-S06:26]** The aggregate losses of Eiffel Auto Insurance are denoted in euro currency and follow a lognormal distribution with $\mu = 8$ and $\sigma = 2$.

Given that 1 euro = 1.3 dollars, which set of lognormal parameters describes the distribution of Eiffel's losses in dollars?

(A) $\mu = 6.15$, $\sigma = 2.26$
(B) $\mu = 7.74$, $\sigma = 2.00$
(C) $\mu = 8.00$, $\sigma = 2.60$
(D) $\mu = 8.26$, $\sigma = 2.00$
(E) $\mu = 10.40$, $\sigma = 2.60$

2.4. **[4B-S90:37]** (2 points) Liability claim severity follows a Pareto distribution with a mean of 25,000 and parameter $\alpha = 3$.

If inflation increases all claims by 20%, the probability of a claim exceeding 100,000 increases by what amount?

(A) Less than 0.02
(B) At least 0.02, but less than 0.03
(C) At least 0.03, but less than 0.04
(D) At least 0.04, but less than 0.05
(E) At least 0.05

Exercises continue on the next page . . .

2.5. **[4B-F97:26]** (3 points) You are given the following:

- In 1996, losses follow a lognormal distribution with parameters μ and σ.
- In 1997, losses follow a lognormal distribution with parameters $\mu + \ln k$ and σ, where k is greater than 1.
- In 1996, $100p\%$ of the losses exceed the mean of the losses in 1997.

Determine σ.

Note: z_p is the $100p^{\text{th}}$ percentile of a normal distribution with mean 0 and variance 1.

(A) $2\ln k$

(B) $-z_p \pm \sqrt{z_p^2 - 2\ln k}$

(C) $z_p \pm \sqrt{z_p^2 - 2\ln k}$

(D) $\sqrt{-z_p \pm \sqrt{z_p^2 - 2\ln k}}$

(E) $\sqrt{z_p \pm \sqrt{z_p^2 - 2\ln k}}$

2.6. **[4B-S94:16]** (1 point) You are given the following:

- Losses in 1993 follow the density function
$$f(x) = 3x^{-4}, \quad x \geq 1,$$
where x = losses in millions of dollars.
- Inflation of 10% impacts all claims uniformly from 1993 to 1994.

Determine the probability that losses in 1994 exceed 2.2 million.

(A) Less than 0.05
(B) At least 0.05, but less than 0.10
(C) At least 0.10, but less than 0.15
(D) At least 0.15, but less than 0.20
(E) At least 0.20

2.7. **[4B-F95:6]** (2 points) You are given the following:

- In 1994, losses follow a Pareto distribution with parameters $\theta = 500$ and $\alpha = 1.5$.
- Inflation of 5% impacts all losses uniformly from 1994 to 1995.

What is the median of the portion of the 1995 loss distribution above 200?

(A) Less than 600
(B) At least 600, but less than 620
(C) At least 620, but less than 640
(D) At least 640, but less than 660
(E) At least 660

Exercises continue on the next page . . .

2.8. [**CAS3-S04:34**] Claim severities are modeled using a continuous distribution and inflation impacts claims uniformly at an annual rate of i.

Which of the following are true statements regarding the distribution of claim severities after the effect of inflation?

1. An Exponential distribution will have scale parameter $(1+i)\theta$
2. A 2-parameter Pareto distribution will have scale parameters $(1+i)\alpha$ and $(1+i)\theta$.
3. A Paralogistic distribution will have scale parameter $\theta/(1+i)$

(A) 1 only (B) 3 only (C) 1 and 2 only (D) 2 and 3 only (E) 1, 2, and 3

2.9. [**CAS3-F05:21**] Losses during the current year follow a Pareto distribution with $\alpha = 2$ and $\theta = 400{,}000$. Annual inflation is 10%.

Calculate the ratio of the expected proportion of claims that will exceed \$750,000 next year to the proportion of claims that exceed \$750,000 this year.

(A) Less than 1.105
(B) At least 1.105, but less than 1.115
(C) At least 1.115, but less than 1.125
(D) At least 1.125, but less than 1.135
(E) At least 1.135

2.10. [**4B-S99:17**] You are given the following:

- In 1998, claim sizes follow a Pareto distribution with parameters θ (unknown) and $\alpha = 2$.
- Inflation of 6% affects all claims uniformly from 1998 to 1999.
- r is the ratio of the proportion of claims that exceed d in 1999 to the proportion of claims that exceed d in 1998.

Determine the limit of r as d goes to infinity.

(A) Less than 1.05
(B) At least 1.05, but less than 1.10
(C) At least 1.10, but less than 1.15
(D) At least 1.15, but less than 1.20
(E) At least 1.20

Exercises continue on the next page . . .

2.11. **[4B-F94:28]** (2 points) You are given the following:

- In 1993, the claim amounts for a certain line of business were normally distributed with mean $\mu = 1000$ and variance $\sigma^2 = 10{,}000$;

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \quad -\infty < x < \infty, \quad \mu = 1000,\ \sigma = 100.$$

- Inflation of 5% impacted all claims uniformly from 1993 to 1994.

What is the distribution for claim amounts in 1994?

(A) No longer a normal distribution
(B) Normal with $\mu = 1000$ and $\sigma = 102.5$.
(C) Normal with $\mu = 1000$ and $\sigma = 105.0$.
(D) Normal with $\mu = 1050$ and $\sigma = 102.5$.
(E) Normal with $\mu = 1050$ and $\sigma = 105.0$.

2.12. **[4B-S93:11]** (1 point) You are given the following:

(i) The underlying distribution for 1992 losses is given by a lognormal distribution with parameters $\mu = 17.953$ and $\sigma = 1.6028$.
(ii) Inflation of 10% impacts all claims uniformly the next year.

What is the underlying loss distribution after one year of inflation?

(A) Lognormal with $\mu' = 19.748$ and $\sigma' = 1.6028$.
(B) Lognormal with $\mu' = 18.048$ and $\sigma' = 1.6028$.
(C) Lognormal with $\mu' = 17.953$ and $\sigma' = 1.7631$.
(D) Lognormal with $\mu' = 17.953$ and $\sigma' = 1.4571$.
(E) No longer a lognormal distribution

2.13. X follows an exponential distribution with mean 10.

Determine the mean of X^4.

2.14. You are given

(i) X is exponential with mean 2.
(ii) $Y = X^{1.5}$.

Calculate $\mathbf{E}[Y^2]$.

2.15. X follows a gamma distribution with parameters $\alpha = 2.5$ and $\theta = 10$.

$Y = 1/X$.

Evaluate $\operatorname{Var}(Y)$.

Solutions

2.1. Looking up the tables for the inverse gamma distribution, we see that the mode is $\frac{\theta}{\alpha+1}$ and the mean is $\frac{\theta}{\alpha-1}$, so

$$\frac{\theta}{\alpha+1} = 4$$

$$\frac{\theta}{\alpha-1} = 6$$

Dividing the second line into the first,

$$\begin{aligned}\frac{\alpha-1}{\alpha+1} &= \frac{4}{6}\\ \alpha = 5 &\qquad \theta = 24\end{aligned}$$

Then (γ_1 is the coefficient of skewness).

$$\begin{aligned}
\mathbf{E}\left[X^2\right] &= \frac{\theta^2}{(\alpha-1)(\alpha-2)} = \frac{24^2}{12} = 48\\
\operatorname{Var}(X) &= 48 - 6^2 = 12\\
\mathbf{E}\left[X^3\right] &= \frac{\theta^3}{(\alpha-1)(\alpha-2)(\alpha-3)} = \frac{24^3}{24} = 576\\
\gamma_1 &= \frac{576 - 3(48)(6) + 2(6^3)}{12^{1.5}}\\
&= \frac{144}{12^{1.5}} = \sqrt{12} = \boxed{\mathbf{3.4641}} \qquad \textbf{(E)}
\end{aligned}$$

2.2. The mean in 2022 is $\theta/3$. By definition, the 65th percentile is the number π_{65} such that $F(\pi_{65}) = 0.65$, so $F(\theta/3) = 0.65$ for the 2023version of F. In 2023, F is two-parameter Pareto with inflated parameter $\theta' = (1+r)\theta$ and $\alpha = 4$, so

$$\begin{aligned}
1 - \left(\frac{\theta'}{\theta' + (\theta/3)}\right)^4 &= 0.65\\
\frac{(1+r)\theta}{(1+r)\theta + \theta/3} &= \sqrt[4]{0.35}\\
\frac{1+r}{4/3+r} &= \sqrt[4]{0.35}\\
r(1-\sqrt[4]{0.35}) &= \frac{4}{3}\sqrt[4]{0.35} - 1\\
r &= \frac{(4/3)\sqrt[4]{0.35} - 1}{1-\sqrt[4]{0.35}} = \boxed{\mathbf{0.1107}}
\end{aligned}$$

2.3. To scale a lognormal, you leave σ and add the logarithm of the scale to μ. So already, B and D are the only possibilities, and the question is whether you add or subtract $\ln 1.3$. But clearly the mean in dollars is higher than the mean in euros, so you add $\ln 1.3$ and get **(D)**.

2.4. Let X be the original variable, $Z = 1.2X$. Since the mean is 25,000, the parameter θ is $25{,}000(\alpha-1) = 50{,}000$.

$$\begin{aligned}
\Pr(X > 100{,}000) &= \left(\frac{50}{150}\right)^3 = \frac{1}{27}\\
\Pr(Z > 100{,}000) &= \left(\frac{60}{160}\right)^3 = \frac{27}{512}\\
\frac{27}{512} - \frac{1}{27} &= \boxed{\mathbf{0.0157}} \qquad \textbf{(A)}
\end{aligned}$$

2.5. The key is to understand (iii). If (for example) 30% of losses exceed \$10000, what percentage does not exceed \$10000? (Answer: 70%) And what percentile of the distribution of losses is \$10000? (Answer: 70$^{\text{th}}$). So statement (iii) is saying that the $100(1-p)^{\text{th}}$ percentile of losses in 1996 equals the mean of losses in 1997. Got it?

The mean of 1997 losses is $\exp(\mu + \ln k + \frac{\sigma^2}{2})$. The $100(1-p)$th percentile is $\exp(\mu - z_p\sigma)$. So:

$$\mu - z_p\sigma = \mu + \ln k + \frac{\sigma^2}{2}$$
$$\frac{\sigma^2}{2} + \sigma z_p + \ln k = 0$$
$$\sigma = \boxed{\boldsymbol{-z_p \pm \sqrt{z_p^2 - 2\ln k}}} \qquad \textbf{(B)}$$

Notice that p must be less than 0.5, by the following reasoning. In general, the median of a lognormal (e^{μ}) is less than (or equal to, if $\sigma = 0$) the mean ($e^{\mu+\sigma^2/2}$), so the median of losses in 1996 is no more than the mean of losses in 1996, which in turn is less than the mean of losses in 1997 since $k > 1$, so $100p$ must be less than 50. Since p is less than 0.5, it follows that z_p will be negative, and σ is therefore positive, as it should be.

2.6. We recognize the 1993 distribution as a single-parameter Pareto with $\theta = 1$, $\alpha = 3$. The inflated parameters are $\theta = 1.1$, $\alpha = 3$. $\left(\frac{1.1}{2.2}\right)^3 = \boxed{\mathbf{0.125}}$. **(C)**

2.7. Let X be the inflated variable, with $\theta = 525$, $\alpha = 1.5$. $\Pr(X > 200) = \left(\frac{525}{525+200}\right)^{1.5} = 0.6162$. Let F be the original distribution function, F^* the distribution of $X \mid X > 200$. Then $F(200) = 1 - 0.6162 = 0.3838$ and

$$F^*(x) = \Pr(X \le x \mid X > 200) = \frac{\Pr(200 < X \le x)}{\Pr(X > 200)} = \frac{F(x) - F(200)}{1 - F(200)}$$

So to calculate the median, we set $F^*(x) = 0.5$, which means

$$\frac{F(x) - F(200)}{1 - F(200)} = 0.5$$
$$\frac{F(x) - 0.3838}{0.6162} = 0.5$$
$$F(x) = 0.5(0.6162) + 0.3838 = 0.6919$$

We must find x such that $F(x) = 0.6919$.

$$1 - \left(\frac{525}{525+x}\right)^{1.5} = 0.6919$$
$$\frac{525}{525+x} = 0.4562$$
$$\frac{525 - 525(0.4562)}{0.4562} = x$$
$$x = \boxed{\mathbf{625.87}} \qquad \textbf{(C)}$$

2.8. All the distributions are parameterized so that θ is the scale parameter and is multiplied by $1+i$; no other parameters change, and you should never divide by $1+i$. Therefore $\boxed{\textbf{only 1}}$ is correct. **(A)**

2.9. The Pareto is a scale distribution with scale parameter θ, so annual inflation of 10% increases θ by 10%, making it 440,000. The proportion of claims above 750,000 is $S(750{,}000) = \left(\frac{\theta}{\theta+750{,}000}\right)^{\alpha}$. Hence, the proportion this year is $\left(\frac{400{,}000}{400{,}000+750{,}000}\right)^2 = 0.120983$ and the proportion next year is $\left(\frac{440{,}000}{440{,}000+750{,}000}\right)^2 = 0.136714$. The ratio is $\frac{0.136714}{0.120983} = \boxed{\mathbf{1.1300}}$. **(D)**

2.10. This is:

$$\frac{\left(\frac{1.06\theta}{1.06\theta+d}\right)^2}{\left(\frac{\theta}{\theta+d}\right)^2} = \frac{1.06^2(\theta+d)^2}{(1.06\theta+d)^2} \longrightarrow 1.06^2 = \boxed{\mathbf{1.1236}}. \qquad \textbf{(C)}$$

2.11. If X is normal, then $aX + b$ is normal as well. In particular, $1.05X$ is normal. So the distribution of claims after 5% uniform inflation is normal.

For any distribution, multiplying the distribution by a constant multiplies the mean and standard deviation by that same constant. Thus in this case, the new mean is 1050 and the new standard deviation is 105. **(E)**

2.12. Add $\ln 1.1$ to μ: $17.953 + \ln 1.1 = 18.048$. σ does not change. **(B)**

2.13. The k^{th} moment for an exponential is given in the tables:

$$\mathbf{E}[X^k] = k!\theta^k$$

for $k = 4$ and the mean $\theta = 10$, this is $4!(10^4) = \boxed{\mathbf{240{,}000}}$.

2.14. While Y is Weibull, you don't need to know that. It's simpler to use $Y^2 = X^3$ and look up the third moment of an exponential.

$$\mathbf{E}[X^3] = 3!\theta^3 = 6(2^3) = \boxed{\mathbf{48}}$$

2.15. We calculate $\mathbf{E}[Y]$ and $\mathbf{E}[Y^2]$, or $\mathbf{E}[X^{-1}]$ and $\mathbf{E}[X^{-2}]$. Note that the special formula in the tables for integral moments of a gamma, $\mathbf{E}[X^k] = \theta^k(\alpha + k - 1)\cdots\alpha$ only applies when k is a *positive* integer, so it cannot be used for the -1 and -2 moments. Instead, we must use the general formula for moments given in the tables,

$$\mathbf{E}[X^k] = \frac{\theta^k\Gamma(\alpha+k)}{\Gamma(\alpha)}$$

For $k = -1$, this is

$$\mathbf{E}[X^{-1}] = \frac{\theta^{-1}\Gamma(\alpha-1)}{\Gamma(\alpha)} = \frac{1}{\theta(\alpha-1)}$$

since $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$. For $k = -2$,

$$\mathbf{E}[X^{-2}] = \frac{1}{\theta^2(\alpha-1)(\alpha-2)}$$

Therefore,

$$\mathrm{Var}(Y) = \frac{1}{10^2(1.5)(0.5)} - \left(\frac{1}{10(1.5)}\right)^2 = \boxed{\mathbf{0.00888889}}$$

Part VII

Practice Exams

It's now time to see how well you would do on an exam.

The distribution of topics on these 12 exams is approximately the distribution specified by the syllabus. The difficulty of the exams is random; all exams have questions of varying levels of difficulty.

Practice Exam 1

1. For a health insurance coverage, there are two types of policyholders.

75% of policyholders are healthy. Annual claim costs for those policyholders have mean 2,000 and variance 10,000,000.

25% of policyholders are in bad health. Annual claim costs for those policyholders have mean 10,000 and variance 50,000,000.

Calculate the variance of annual claim costs for a policyholder selected at random.

(A) 20,000,000 (B) 28,000,000 (C) 32,000,000 (D) 44,000,000 (E) 48,000,000

2. Bill Driver and Jane Motorist are involved in an automobile accident. Jane Motorist's car is totally destroyed. Its value before the accident was 8000, and the scrap metal after the accident is worth 500. Bill Driver is at fault.

Big Insurance Company insures Jane Motorist. Jane has liability insurance with a 100,000 limit and collision insurance with a 1000 deductible.

Standard Insurance Company insures Bill Driver. Bill has liability insurance with a 50,000 limit and collision insurance with a 500 deductible.

Jane files a claim with Big Insurance Company and receives 7000.

Calculate the net amount that Big Insurance Company receives (net of payment of subrogation proceeds to Jane) from subrogation.

(A) 6000 (B) 6500 (C) 7000 (D) 7500 (E) 8000

3. An excess of loss catastrophe reinsurance treaty covers the following layers, expressed in millions:

80% of 100 excess of 100
85% of 200 excess of 200
90% of 400 excess of 400

Calculate the reinsurance payment for a catastrophic loss of 650 million.

(A) 225 million (B) 475 million (C) 495 million (D) 553 million (E) 585 million

4. A rate filing for six-month policies will be effective starting October 1, CY6 for 2 years.
Losses for this rate filing were incurred in AY1 and the amount paid through 12/31/AY4 is 3,500,000.
Trend is at annual effective rate of 6.5%.
Loss development factors are:

$$3/2\text{: } 1.50, \qquad 4/3\text{: } 1.05, \qquad \infty/4\text{: } 1.05$$

Calculate trended and developed losses for AY1.

(A) Less than 5,600,000
(B) At least 5,600,000, but less than 5,700,000
(C) At least 5,700,000, but less than 5,800,000
(D) At least 5,800,000, but less than 5,900,000
(E) At least 5,900,000

5. You are given

Accident Year	Cumulative Payments through Development Year 0	1	2	Earned Premium
AY1	25,000	41,000	48,000	120,000
AY2	30,000	45,000		140,000
AY3	33,000			150,000

The loss ratio is 60%.
Calculate the loss reserve using the loss ratio method.

(A) 100,000 (B) 105,000 (C) 110,000 (D) 115,000 (E) 120,000

6. You are given the following observations:

2, 10, 28, 64, 100

The observations are fitted to an inverse exponential distribution using maximum likelihood.
Determine the resulting estimate of the mode.

(A) 3.2 (B) 3.4 (C) 3.6 (D) 3.8 (E) 4.0

7. You own 100 shares of a stock whose current price is 42. You would like to hedge your downside exposure by buying 100 6-month European put options with a strike price of 40. You are given:

(i) The Black-Scholes-Merton framework is assumed.
(ii) The continuously compounded risk-free interest rate is 5%.
(iii) The stock pays no dividends.
(iv) The stock follows a lognormal process with $\mu = 0.06$ and $\sigma = 0.22$.

Determine the cost of the put options.

(A) 121 (B) 123 (C) 125 (D) 127 (E) 129

Exam questions continue on the next page . . .

8. At a company, the number of sick days taken by each employee in a year follows a Poisson distribution with mean λ. Over all employees, the distribution of λ has the following density function:

$$f(\lambda) = \frac{\lambda e^{-\lambda/3}}{9}$$

Calculate the probability that an employee selected at random will take more than 2 sick days in a year.

(A) 0.59 (B) 0.62 (C) 0.66 (D) 0.70 (E) 0.74

9. For loss size X, you are given:

x	$\mathbf{E}[X \wedge x]$
1000	400
2000	700
3000	900
4000	1000
5000	1100
∞	2500

An insurance coverage has an ordinary deductible of 2000.

Calculate the loss elimination ratio after 100% inflation if the deductible is not changed.

(A) 0.08 (B) 0.14 (C) 0.16 (D) 0.20 (E) 0.40

10. In a study on loss sizes on automobile liability coverage, you are given:

(i) 5 observations $x_1, \ldots, x_5$ from a plan with no deductible and a policy limit of 10,000.
(ii) 5 observations at the limit from a plan with no deductible and a policy limit of 10,000.
(iii) 5 observations $y_1, \ldots, y_5$ from a plan with a deductible of 10,000 and no policy limit.

Which of the following is the likelihood function for this set of observations?

(A) $\prod_{i=1}^{5} f(x_i) \prod_{i-1}^{5} f(y_i)$

(B) $\bigl(1 - F(10{,}000)\bigr)^5 \prod_{i=1}^{5} f(x_i) \prod_{i-1}^{5} f(y_i)$

(C) $\dfrac{\bigl(1 - F(10{,}000)\bigr)^5 \prod_{i=1}^{5} f(x_i) \prod_{i-1}^{5} f(y_i)}{\bigl(F(10{,}000)\bigr)^5}$

(D) $\dfrac{\prod_{i=1}^{5} f(x_i) \prod_{i=1}^{5} f(y_i)}{\bigl(1 - F(10{,}000)\bigr)^5}$

(E) $\dfrac{\bigl(F(10{,}000)\bigr)^5 \prod_{i=1}^{5} f(x_i) \prod_{i-1}^{5} f(y_i)}{\bigl(1 - F(10{,}000)\bigr)^5}$

Exam questions continue on the next page . . .

11. A study on claim sizes produced the following results:

Claim size	Number	Deductible	Limit
500	4	None	10,000
1000	4	500	None
2000	3	500	None
5000	2	None	10,000
At limit	5	None	10,000

A single-parameter Pareto with $\theta = 400$ is fitted to the data using maximum likelihood.

Determine the estimate of α.

(A) 0.43 (B) 0.44 (C) 0.45 (D) 0.61 (E) 0.62

12. You are given:

Accident Year	Incurred Losses through Development Year 0	1	2	3	4	Earned Premium
AY1	7,800	8,900	9,500	11,000	11,000	16,000
AY2	9,100	9,800	10,500	10,800		20,000
AY3	8,600	9,500	10,100			23,000
AY4	9,500	10,000				24,000
AY5	10,700					25,000

The expected loss ratio is 0.7.

Losses mature at the end of 3 years.

Calculate the IBNR reserve using the Bornhuetter-Ferguson method with volume-weighted average loss development factors.

(A) 7,100 (B) 7,200 (C) 7,300 (D) 7,400 (E) 7,500

13. On an automobile liability coverage, annual claim counts follow a negative binomial distribution with mean 0.2 and variance 0.3. Claim sizes follow a two-parameter Pareto distribution with $\alpha = 3$ and $\theta = 10$. Claim counts and claim sizes are independent.

Calculate the variance of annual aggregate claim costs.

(A) 22.5 (B) 25.0 (C) 27.5 (D) 32.5 (E) 35.0

14. A minor medical insurance coverage has the following provisions:

(i) Annual losses in excess of 10,000 are not covered by the insurance.
(ii) The policyholder pays the first 1,000 of annual losses.
(iii) The insurance company pays 60% of the excess of annual losses over 1,000, after taking into account the limitation mentioned in (i).

Annual losses follow a two-parameter Pareto distribution with $\alpha = 4$ and $\theta = 8000$.

Calculate expected annual payments for one policyholder under this insurance.

(A) 983 (B) 1004 (C) 1025 (D) 1046 (E) 1067

Exam questions continue on the next page . . .

15. A company has 100 shares of ABC stock. The current price of ABC stock is 30. ABC stock pays no dividends. The company would like to guarantee its ability to sell the stock at the end of six months for at least 28. European call options on the same stock expiring in 6 months with exercise price 28 are available for 4.10. The continuously compounded risk-free interest rate is 5%.

Determine the cost of the hedge.

(A) 73 (B) 85 (C) 99 (D) 126 (E) 141

16. Let X be the random variable with distribution function

$$F_X(x) = 1 - 0.6e^{-x/10} - 0.4e^{-x/20}$$

Calculate $\mathrm{TVaR}_{0.95}(X)$.

(A) 59 (B) 60 (C) 61 (D) 62 (E) 63

17. For a discrete probability distribution in the $(a, b, 0)$ class, you are given

(i) $p_2 = 0.0768$
(ii) $p_3 = p_4 = 0.08192$

Determine p_0.

(A) 0.02 (B) 0.03 (C) 0.04 (D) 0.05 (E) 0.06

18. Losses on an insurance coverage follow a distribution with density function

$$f(x) = \frac{3}{100^3}(100 - x)^2 \qquad 0 \le x \le 100$$

Losses are subject to an ordinary deductible of 15.

Calculate the loss elimination ratio.

(A) 0.46 (B) 0.48 (C) 0.50 (D) 0.52 (E) 0.54

19. A reinsurance company offers a stop-loss reinsurance contract that pays the excess of annual aggregate losses over 3.

You are given:

(i) Loss counts follow a binomial distribution with $m = 3$ and $q = 0.2$.
(ii) Loss sizes have the following distribution:

Size	Probability
1	0.6
2	0.2
3	0.1
4	0.1

Calculate the expected annual payment under the stop-loss reinsurance contract.

(A) 0.09 (B) 0.10 (C) 0.11 (D) 0.12 (E) 0.13

Exam questions continue on the next page . . .

20. Annual claim frequency follows a Poisson distribution. Loss sizes follow a Weibull distribution with $\tau = 0.5$.

Full credibility for aggregate loss experience is granted if the probability that aggregate losses differ from expected by less than 6% is 95%.

Determine the number of expected claims needed for full credibility.

(A) 6403 (B) 6755 (C) 7102 (D) 7470 (E) 7808

Solutions to the above questions begin on page 627.

Auto-Grade My Exam (Click Here)

Appendices

Appendix A. Solutions to the Practice Exams

Answer Key for Practice Exam 1

1	C	
2	B	
3	B	
4	D	
5	E	
6	D	
7	E	
8	E	
9	C	
10	A	
11	C	
12	D	
13	A	
14	A	
15	E	
16	E	
17	C	
18	B	
19	E	
20	A	

Reset the Exam

Practice Exam 1

1. **[Section 4.1]** You may either use the conditional variance formula (equation (4.1)), or compute first and second moments.

With the conditional variance formula, let I be the indicator of whether the policyholder is healthy or in bad health. Let X be annual claim counts then

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{Var}_I\left(\mathbf{E}_X[X \mid I]\right) + \mathbf{E}_I\left[\mathrm{Var}_X(X \mid I)\right]\\ &= \mathrm{Var}_I(2000, 10{,}000) + \mathbf{E}_I[10{,}000{,}000, 50{,}000{,}000]\end{aligned}$$

where $\mathrm{Var}_I(2{,}000, 10{,}000)$ means the variance of a random variable that is 2,000 with probability 0.75 and 10,000 with probability 0.25. By the Bernoulli shortcut, the variance of such a random variable is

$$(0.75)(0.25)(10{,}000 - 2{,}000)^2 = 12{,}000{,}000$$

$\mathbf{E}_I[10{,}000{,}000, 50{,}000{,}000]$ means the expected value of a random variable that is 10,000,000 with probability 0.75 and 50,000,000 with probability 0.25. The expected value of such a random variable is

$$0.75(10{,}000{,}000) + 0.25(50{,}000{,}000) = 20{,}000{,}000$$

Adding up the variance of the mean and the mean of the variances, we get $\mathrm{Var}(X) = 12{,}000{,}000 + 20{,}000{,}000 = \boxed{\mathbf{32{,}000{,}000}}$. **(C)**

With first and second moments, the overall first moment of annual claim counts is

$$0.75(2{,}000) + 0.25(10{,}000) = 4{,}000$$

The overall second moment of annual claim counts is the weighted average of the individual second moments, and the second moment for each type of driver is the variance plus the mean squared.

$$0.75(10{,}000{,}000 + 2{,}000^2) + 0.25(50{,}000{,}000 + 10{,}000^2) = 48{,}000{,}000$$

The overall variance of claim counts is $48{,}000{,}000 - 4{,}000^2 = \boxed{\mathbf{32{,}000{,}000}}$. **(C)**

2. **[Lesson 5]** Big Insurance Company pays Jane 7000 and receives the scrap metal, for a net loss of 6500. That is the amount that it gets upon subrogration. The remaining 1000 of the subrogation is paid to Jane. **(B)**

3. **[Lesson 13]** The layers are 100–200, 200–400, and 400–800. The amount of the catastrophic loss in each of those layers is 100, 200, and 250 respectively. The reinsurance pays $0.8(100) + 0.85(200) + 0.9(250) = \boxed{\textbf{475 million}}$. **(B)**

4. **[Lesson 9]** Average date of sale of these policies is 10/1/CY7 and average date of accident is 3 months later, or 1/1/CY8. Trend is from 7/1/CY1 through 1/1/CY8, or 6.5 years. Paid data is for year 3, so future development is $(1.05)(1.05) = 1.1025$. Trended and developed losses are $3{,}500{,}000(1.1025)(1.065^{6.5}) = \boxed{\mathbf{5{,}810{,}575}}$. **(D)**

5. **[Section 7.2]** Projected losses based on the loss ratio are

$$0.6(120{,}000 + 140{,}000 + 150{,}000) = 246{,}000$$

Paid to date is $48{,}000 + 45{,}000 + 33{,}000 = 126{,}000$. The reserve is $246{,}000 - 126{,}000 = \boxed{\mathbf{120{,}000}}$. **(E)**

6. **[Lesson 23]** The likelihood function, ignoring the multiplicative constant $1/\prod x_i^2$, is

$$L(\theta) = \theta^5 e^{-\theta \sum 1/x_i}$$